such set of conjugate subgroups is that it contains subgroups of degree n and of index n with respect to which it can be represented as a transitive substitution group which is identical with G. In particular, *if a given abstract group is simply isomorphic with only one group in a complete list of transitive groups of degree n then this transitive group has outer isomorphisms if it contains a subgroup of degree n and of index n which is non-invariant and does not involve any invariant subgroup of the entire group besides the identity.* From this special but useful theorem it follows directly that the symmetric group of degree 6 has outer isomorphisms. Various writers established this fact by somewhat laborious special methods.[1] It also follows directly from this theorem that the largest imprimitive groups on six letters which involve 2 and 3 systems of imprimitivity, respectively, have isomorphisms which cannot be obtained by transforming these groups by substitutions on their own letters.

While the group of inner isomorphisms is an invariant subgroup of I whenever it does not coincide with I it should not be inferred that the subgroup of I which corresponds to all the automorphisms of G which can be obtained by transforming G by substitutions on its own letters is always an invariant subgroup of I. In fact, this is not the case when G is the generalized dihedral group of order 16 involving the abelian group of type (2, 1) represented as a transitive group of degree 8. When G admits automorphisms in which G_1 corresponds to a subgroup of degree n the conjugates of G_1 are transformed under I according to an imprimitive substitution group. One of the systems of imprimitivity of this group is composed of the conjugates of G_1 under G.

[1] Cf. O. Hölder, *Mathematische Annalen*, **46**, 1895 (345); W. Burnside, *Theory of Groups of Finite Order*, 1911, p. 209.

EINSTEIN STATIC FIELDS ADMITTING A GROUP G_2 OF CONTINUOUS TRANSFORMATIONS INTO THEMSELVES

BY L. P. EISENHART

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY

Communicated by O. Veblen, November 3, 1921

1. For static phenomena in the Einstein theory the linear element of the space-time continuum can be taken in the form $V^2dx_0{}^2-ds^2$, where

$$ds^2 = \Sigma\, a_{ik}\, dx_i\, dx_k \ (i, k = 1, 2, 3) \qquad (1)$$

is the linear element of the physical space S, and the functions V and a_{ik} are independent of x_0, the coördinate of time. In this paper we determine the functions V and a_{ik} satisfying the Einstein equations $B_{ik} = O$ and such that the space S admits a continuous group G_2 of transformations into itself. Bianchi[1] has shown that any 3-space admitting such a group is of one of two types:

(I) $$ds^2 = dx_1^2 + \alpha dx_2^2 + 2\beta dx_2 dx_3 + \gamma dx_3^2,$$

α, β, γ being functions of x_1 alone, the operators of the group being

$$X_1 = \partial/\partial x_2, \quad X_2 = \partial/\partial x_3;$$

(II) $$ds^2 = dx_1^2 + \alpha dx_2^2 + 2(\beta - \alpha x_2)\, dx_2 dx_3 + (\alpha x_2^2 - 2\beta x_2 + \gamma) dx_3^2,$$

α, β, γ being functions of x_1 alone, the operators of the group being

$$X_1 = \partial/\partial x_3, \quad X_2 = e^{x_3}\partial/\partial x_2.$$

2. By definition

$$B_{ik} = \sum_0^3 {}_h \left\{ih, hk\right\} = \sum_0^3 {}_h \left[\frac{\partial}{\partial x_k}\begin{Bmatrix} ih \\ h \end{Bmatrix} - \frac{\partial}{\partial x_h}\begin{Bmatrix} ik \\ h \end{Bmatrix} + \sum_0^3 {}_e\left(\begin{Bmatrix} ih \\ e \end{Bmatrix}\begin{Bmatrix} ke \\ h \end{Bmatrix} - \begin{Bmatrix} lh \\ h \end{Bmatrix}\begin{Bmatrix} ik \\ e \end{Bmatrix}\right)\right], \tag{2}$$

where the Christoffel quantities are calculated with respect to the linear element of the space-time continuum. When this linear element is taken in the form $V^2dx_0^2 - ds^2$ as in § 1, the equations $B_{ik} = 0$ reduce to the six equations[2]

$$\frac{\partial^2 V}{\partial x_i \partial x_k} = \sum_1^3 {}_e \begin{Bmatrix} ik \\ e \end{Bmatrix}\frac{\partial V}{\partial x_e} - \overline{B}_{ik} V \; (i, k = 1, 2, 3), \tag{3}$$

and

$$\sum_{i,k} a^{ik}\overline{B}_{ik} = 0, \tag{4}$$

where a^{ik} is the cofactor of a_{ik} in the determinant of the quantities a_{ik} divided by this determinant; the Christoffel symbols are calculated with respect to (1), and

$$\overline{B}_{ik} = \sum_1^3 {}_h \left\{ih, hk\right\} = \sum_1^3 {}_h \sum_1^3 {}_l\, a^{hl}\left(il, hk\right). \tag{5}$$

By assuming that S is referred to a triply-orthogonal system, so that the linear element may be written $ds^2 = \sum_1^3 {}_i\, a_i dx_i^2$ (which is possible in any 3-space[3]), and making use of the relations

$$(il,hk) = -(li, hk) = -(il, kh) = (li, kh),$$

we find that when $\overline{B}_{ik} = 0$, then all the functions (il, hk) are equal to zero. This being the condition that the space S is euclidean, we have the theorem:

A necessary and sufficient condition that a 3-space be euclidean is that the functions $\overline{B}_{ik} = 0$.

3. If we put

$$\delta = \alpha\gamma - \beta^2,\; A = \beta'\gamma - \beta\gamma',\; B = \gamma'\alpha - \gamma\alpha',\; C = \alpha'\beta - \alpha\beta',$$

(where primes indicate differentiation with respect to x_1) we find that all of the Christoffel quantities for the form (I) vanish except the following:

$$\left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\} = -\frac{\alpha'}{2},\; \left\{ \begin{matrix} 23 \\ 1 \end{matrix} \right\} = -\frac{\beta'}{2},\; \left\{ \begin{matrix} 33 \\ 1 \end{matrix} \right\} = -\frac{\gamma'}{2},\; \left\{ \begin{matrix} 13 \\ 2 \end{matrix} \right\} = \frac{A}{2\delta},$$

$$\left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\} = \frac{1}{4\delta}\left(\delta' - B\right),\; \left\{ \begin{matrix} 12 \\ 3 \end{matrix} \right\} = -\frac{C}{2\delta},\; \left\{ \begin{matrix} 13 \\ 3 \end{matrix} \right\} = \frac{1}{4\delta}\left(\delta' + B\right).$$

Also we find

$$\overline{B}_{11} = \frac{\delta''}{2\delta} - \frac{1}{4}\frac{\delta'^2}{\delta^2} - \frac{1}{2\delta}\left(\alpha'\gamma' - \beta'^2\right),\; \overline{B}_{12} = \overline{B}_{13} = 0,$$

$$\overline{B}_{22} = \frac{1}{2}\alpha'' - \frac{\alpha'}{4}\frac{\delta'}{\delta} + \frac{\alpha}{2\delta}\left(\alpha'\gamma' - \beta'^2\right),$$

$$\overline{B}_{23} = \frac{1}{2}\beta'' - \frac{\beta'}{4}\frac{\delta'}{\delta} + \frac{\beta}{2\delta}\left(\alpha'\gamma' - \beta'^2\right),$$

$$\overline{B}_{33} = \frac{1}{2}\gamma'' - \frac{\gamma'}{4}\frac{\delta'}{\delta} + \frac{\gamma}{2\delta}\left(\alpha'\gamma' - \beta'^2\right).$$

Equation (4) reduces to

$$2\,\alpha\,\gamma'' + 2\,\alpha''\,\gamma + 3\left(\alpha'\,\gamma' - \beta'^2\right) - 4\,\beta\,\beta'' - \frac{\delta'^2}{\delta} = 0. \qquad (6)$$

The problem reduces to the integration of (3), when α, β, γ are subject to the condition (6). We separate the discussion into the four cases, when V is independent of x_2 and x_3; of either x_2 or x_3; of x_1; involves all three variables. Expressing the conditions of integrability of (3) we are led to linear partial differential equations of the first order and these are ultimately solved. The first and second cases are the only ones giving solutions; in the course of the investigation it is shown that in both cases a change of variables can be made so that $\beta = 0$. The solutions of the first type are:

$$\alpha = x_1^{\frac{2(k+1)}{k^2+k+1}},\; \gamma = x_1^{\frac{2k(k+1)}{k^2+k+1}},\; V = x_1^{-\frac{k}{k^2+k+1}}, \qquad (7)$$

where k is any constant. Solutions of this kind have been found by Kasner.[4] For the second case the linear element of S may be written

$$ds^2 = \frac{d\alpha^2}{b\,\alpha + a\,\alpha^{1/2}} + \alpha\,dx_2^2 + \left(b + a\,\alpha^{-\frac{1}{2}}\right)d\,x_3^2, \qquad (8)$$

and

$$V = \alpha^{1/2}\varphi(x_2), \qquad \frac{d^2\varphi}{dx_2^2} + b\varphi = 0, \qquad (9)$$

where a and b are constants. This is the solution found by Levi-Civita[5] and called the *quadrantal* solution.

4. For the linear element (II) the expressions for the Christoffel symbols which are different from zero are:

$$\left\{ {22 \atop 1} \right\} = -\frac{\alpha'}{2}, \left\{ {23 \atop 1} \right\} = \frac{1}{2}\left(\alpha' x_2 - \beta'\right),$$

$$\left\{ {33 \atop 1} \right\} = -\frac{1}{2}\left(\alpha' x_2^2 - 2\beta' x_2 + \gamma'\right),$$

$$\left\{ {12 \atop 2} \right\} = \frac{1}{4\delta}\left(-2Cx_2 - B + \delta'\right), \left\{ {13 \atop 2} \right\} = \frac{1}{2\delta}\left(Cx_2^2 + Bx_2 + A\right),$$

$$\left\{ {22 \atop 2} \right\} = -\frac{\alpha}{\delta}\left(\alpha x_2 - \beta\right), \left\{ {23 \atop 2} \right\} = \frac{(\alpha x_2 - \beta)^2}{\delta},$$

$$\left\{ {33 \atop 2} \right\} = \frac{\beta - \alpha x_2}{\delta}\left(\alpha x_2^2 - 2\beta x_2 + \gamma\right),$$

$$\left\{ {12 \atop 3} \right\} = -\frac{C}{2\delta}, \left\{ {13 \atop 3} \right\} = \frac{1}{4\delta}\left(2Cx_2 + B + \delta'\right), \left\{ {22 \atop 3} \right\} = -\frac{\alpha^2}{\delta},$$

$$\left\{ {23 \atop 3} \right\} = \frac{\alpha(\alpha x_2 - \beta)}{\delta}, \left\{ {33 \atop 3} \right\} = -\frac{(\alpha x_2 - \beta)^2}{\delta}.$$

The expressions for the functions $\overline{B}_{ik}$ are

$$\overline{B}_{11} = \frac{\delta''}{2\delta} - \frac{1}{4}\frac{\delta'^2}{\delta^2} - \frac{1}{2\delta}\left(\alpha'\gamma' - \beta'^2\right), \quad \overline{B}_{12} = \frac{C}{\delta},$$

$$\overline{B}_{13} = -\frac{1}{2\delta}\left(2Cx_2 + B\right),$$

$$\overline{B}_{22} = L, \overline{B}_{23} = M - x_2 L, \quad \overline{B}_{33} = Lx_2^2 - 2Mx_2 + N,$$

where

$$L = \frac{\alpha''}{2} - \frac{\alpha'\delta'}{4\delta} + \frac{\alpha}{2\delta}\left(\alpha'\gamma' - \beta^2\right) + \frac{\alpha^2}{\delta},$$

$$M = \frac{\beta''}{2} - \frac{\beta'}{4}\frac{\delta'}{\delta} + \frac{\beta}{2\delta}\left(\alpha'\gamma' - \beta'^2\right) + \frac{\alpha\beta}{\delta},$$

$$N = \frac{\gamma''}{2} - \frac{\gamma'}{4}\frac{\delta'}{\delta} + \frac{\gamma}{2\delta}\left(\alpha'\gamma' - \beta'^2\right) + \frac{\alpha\gamma}{\delta}.$$

The condition (4) is

$$2\delta'' - \frac{\delta'^2}{\delta} - \left(\alpha'\gamma' - \beta'^2\right) + 4\alpha = 0. \qquad (10)$$

We consider the system of equations (3) as in § 3, taking up separately the five cases where V is independent of x_2 and x_3; of x_2; of x_1; of x_3; involves all three vaiiables. Only the first two cases lead to solutions.

For the first case we have the linear element of S

$$ds^2 = \frac{d\alpha^2}{4(a\alpha^{1/2} - \alpha)} + \alpha dx_2^2 - 2\alpha x_2 dx_2 dx_3 + \alpha(x_2^2 + 1) dx_3^2, \qquad (11)$$

and

$$V = \sqrt{a\,\alpha^{-1/2} - 1},$$

where a is a constant. This is the *longitudinal* solution obtained by Levi-Civita.[5]

For V independent of x_2 two cases arise, according as β can be made equal to zero by a transformation of coördinates, or not.

When $\beta = o$, the functions α and γ must satisfy the three equations

$$\begin{aligned} \alpha'' + \left(1 + \frac{1}{2k}\right)\frac{\alpha'\gamma'}{\gamma} - \left(1 + \frac{1}{k}\right)\frac{\alpha'^2}{2\alpha} + \frac{2\alpha}{\gamma}\left(1 + k\right) = 0, \\ \gamma'' + \left(1 - \frac{1}{k}\right)\frac{\alpha'\gamma'}{2\alpha} + \frac{1}{2k}\frac{\gamma'^2}{\gamma} + 2\left(1 + k^2\right) = 0, \\ 2\,\alpha'\gamma' - \frac{\alpha'^2}{\alpha}\frac{\gamma}{k} + \left(1 + \frac{1}{k}\right)\frac{\gamma'^2\,\alpha}{\gamma} + 4\,\alpha\left(k^2 + k + 1\right) = 0, \end{aligned} \tag{12}$$

where k is any constant. For $k = 1$,

$$\gamma\,\gamma'^2 + 4\,\gamma^2 = 4\,a, \quad \frac{\alpha}{\gamma} = b\left(\frac{\sqrt{a} + \sqrt{a - \gamma^2}}{\gamma}\right)^{\pm\sqrt{3}}, \quad V = e^{x_3}\sqrt{\alpha\gamma} \tag{13}$$

where a and b are constants. Consequently,

$$ds^2 = \frac{4(a - \gamma^2)}{\gamma}\,d\gamma^2 + \alpha\,dx_2^2 - 2\,\alpha\,x_2\,dx_2\,dx_3 + \left(\alpha\,x_2^2 + \gamma\right)dx_3^2. \tag{14}$$

When $k \neq 1$, we solve the second of (12) for α'/α and substitute in the third of (12). The first integral of the resulting equation is

$$\frac{\left(\gamma' + \sqrt{\gamma'^2 + 4\,k\,\gamma}\right)^b}{\left[(1 + k^2)\sqrt{\gamma'^2 + 4\,k\,\gamma} + b\,\gamma'\right]^{1+k^2}} = a\,\gamma^{k + \frac{b}{2}}, \tag{15}$$

where a is an arbitrary constant and $b = \pm(1 - k)\sqrt{1 + k + k^2}$. Then we have

$$\begin{aligned} \alpha^{\frac{1}{k} - 1} = \gamma^{\frac{1'}{k}}\left[(1 + k^2)\sqrt{\gamma'_2 + 4\,k\,\gamma} + b\,\gamma'\right]^2, \\ V = e^{kx_3}\sqrt{\gamma}\left(\frac{\gamma}{\alpha}\right)^{\frac{1}{2k}}. \end{aligned} \tag{16}$$

When $\beta \neq 0$, we have

$$V = \delta^{1/4}e^{-2x_3}, \quad \delta = \alpha\,\gamma - \beta^2, \tag{17}$$

with the following conditions upon α, β and γ:

$$\begin{aligned} \alpha'' - \frac{\alpha'\,\delta'}{4\delta} - \frac{\alpha}{2}\frac{\delta'^2}{\delta^2} - 14\frac{\alpha^2}{\delta} = 0, \\ \beta'\alpha - \beta\alpha' = k\delta^{1/4}, \\ \delta'^2 + 2\delta\,(\alpha'\,\gamma' - \beta'^2) + 24\,\alpha\,\delta = 0, \end{aligned} \tag{18}$$

where k denotes an arbitrary constant.

Now (10) reduces to

$$\delta'' - \frac{1}{4}\frac{\delta'^2}{\delta} + 8\alpha = 0, \tag{19}$$

which is consistent with (18). Eliminating α from the first of (18) and (19), we get

$$\delta'''' - \frac{3}{4}\frac{\delta'\delta'''}{\delta} + \frac{5}{4}\frac{\delta''^2}{\delta} - \frac{21}{64}\frac{\delta'^4}{\delta^3} = 0.$$

If we introduce dependent and independent variables t and θ by

$$\delta = e^4\int e^\theta dx_1, \quad t = e^{-\theta}\frac{d\theta}{dx_1},$$

we get the equation

$$t\frac{d}{d\theta}\left(t\frac{dt}{d\theta}\right) + 7t^2\frac{dt}{d\theta} + 13t\frac{dt}{d\theta} + (2t+5)(3t+5)(t+3) = 0.$$

To the evident solutions of this equation, $t = -5/2$, $t = -5/3$, correspond the solutions:

$$\delta = 25x_1^{8/5}, \quad \alpha = -x_1^{-2/5}, \beta = bx_1^{9/5}, \gamma = -b^2x_1^4 - 25x_1^2,$$

$$\delta = 25x_1^{12/5}, \quad \alpha = -6x_1^{2/5}, \quad \beta = bx_1^{6/5}, \gamma = -\frac{1}{6}(b^2+25)x_1^2,$$

where b is an arbitrary constant. For $t = -3$, we find $\alpha = 0$, which is excluded since we assume that $\delta > 0$ in accordance with the theory.

If we put $t\, dt/d\theta = y$, and take y for dependent variable, the above equation may be replaced by

$$y\frac{dy}{dt} + y\left(7t+13\right) + \left(2t+5\right)\left(3t+5\right)\left(t+3\right) = 0.$$

When a solution of this equation is known, the corresponding functions α, β, γ can be found by quadratures.

5. Making use of the formulas of Bianchi[6] we find the following expressions for the principal curvatures of the spaces (7), (8), (11), (14)

$$-\underline{\overline{K}}_1 = (1+k)\underline{\overline{K}}_2 = \left(1+\frac{1}{k}\right)\underline{\overline{K}}_3 = \frac{k(k+1)^2}{(1+k+k^2)^2}\frac{1}{x_1^2}; \tag{7*}$$

$$\underline{\overline{K}}_1 = -\frac{\underline{\overline{K}}_2}{2} = \underline{\overline{K}}_3 = \frac{a}{2\alpha^{3/2}}; \tag{8*}$$

$$-\frac{\underline{\overline{K}}_1}{2} = \underline{\overline{K}}_2 = \underline{\overline{K}}_3 = \frac{a}{2\alpha^{3/2}}; \tag{11*}$$

$$\underline{\overline{K}}_1 = -\frac{a}{\gamma^3}(1+\sqrt{3}), \underline{\overline{K}}_2 = \frac{2a}{\gamma^3}, \underline{\overline{K}}_3 = \frac{-a}{\gamma^3}(1-\sqrt{3}). \tag{14*}$$

The principal curvatures for $k \pm 1$ in (12) can be obtained explicitly, but their forms are quite involved.

At each point of space the principal curvatures correspond to three directions, mutually perpendicular to one another. When the curves tangent to these directions are the curves of intersection of a triply-orthogonal system of surfaces, the space is called *normal* by Bianchi. All the spaces referred to above are normal. For the cases (7) and (8) the tangents to the curves of intersection x_i = const., x_j = const. are the principal directions.

For the cases of § 4 we put $x_2 = e^{\bar{x}_2 + x_3}$. Then for (11), the curves of intersection of surfaces x_1 = const., $\bar{x}_2$ = const., x_3 = const. have the principal directions. For (14) and the case $k \neq 1$, the principal directions are given by x_1 = const., x_3 = const., and the orthogonal systems of curves on $\bar{x}_2$ = const. defined by

$$\left(\alpha' - \frac{\alpha\gamma'}{\gamma}\right)dx_1^2 + 2\left[\alpha + \frac{1}{4}\alpha'\gamma' - \gamma\left(\frac{\alpha''}{2} - \frac{\alpha'^2}{4\alpha}\right)\right]dx_1\,dx_3 - \gamma\left(\alpha' - \frac{\alpha\gamma'}{\gamma}\right)dx_3^2 = 0.$$

[1] *Mem. Soc. Ital.*, 1896, p. 347.

[2] Levi-Civita, *Rend Lincei* (ser. 5), **26**, 1917, sem. 1 (460).

[3] Bianchi, *Lezioni*, **1**, 377; Cotton, *Ann. Fac. Toul.* (ser. 2), **1**, 1899 (410).

[4] *Science*, **54**, 1921 (305).

[5] *Rend. Lincei* (ser. 5), **27**, 1918, sem. 2 (350).

[6] *Lezioni*, **1**, 354.

GEOMETRIC ASPECTS OF THE ABELIAN MODULAR FUNCTIONS OF GENUS FOUR (II)

By Arthur B. Coble

Department of Mathematics, University of Illinois

Communicated by E. H. Moore, June 21, 1921

8. *The form* $\binom{3\ 2\ 2}{1\ 1\ 1}$.—This form, written symbolically as $(\rho z)\,(rx)(sy)$, where z is a point in S_3, x a point in S_2, and y a point in S_2', has 36 coefficients and therefore $35-15-8-8 = 4$ absolute projective constants. Points x, y determine a plane which becomes indeterminate for six pairs $x, y = p_i\, q_i$; $(i = 1, \ldots, 6)$ which form associated six points. They are the double singular points of a Cremona transformation T of the fifth order between the planes S_x, S_y. A given plane u is determined by ∞^1 pairs x, y which lie, respectively, on the cubic curves, $(\rho\rho'\rho''u)\,(rx)\,(r'x)\,(r''x)\,(ss's'') = 0$, $(\rho\rho'\rho''u)\,(rr'r'')\,(sy)\,(s'y)\,(s''y) = 0$ These curves pass, respectively, through the six points p_i and the six points q_i. Thus the given form is associated with a general cubic surface, $(\rho z)\,(\rho' z)\,(\rho'' z)\,(rr'r'')\,(ss's'') = 0$, with an isolated double-six of lines and separated